Phase-equivalent potentials from supersymmetry

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Abstract. Potentials providing the same phase shifts as a given potential, and the same bound spectrum except possibly for the ground state, are derived from combinations of supersymmetry transformations. The suppression of the ground state or the addition of a new ground state introduces an additional r^{-2} singularity in the potential in agreement with a generalised Levinson theorem. Different properties of the phase-equivalent potentials and of their wavefunctions are established, discussed and illustrated by an example.

1. Introduction

The application of supersymmetry to Schrödinger quantum mechanics (Witten 1981, Andrianov *et al* 1984, Sukumar 1985a, b) has shed new light on the problem of constructing phase-equivalent potentials (Bargmann (1949); see Sukumar (1985b) for further references to the inverse scattering problem). The elementary technique involved in their determination is also known as a Darboux transformation (Deift and Trubowitz 1979). Supersymmetric partners of a radial Hamiltonian with a given central potential possess the same spectrum except for possible suppression of the ground state or addition of a bound state below it (Sukumar 1985b). However, according to the Levinson theorem, non-singular potentials whose numbers of bound states differ cannot provide equal phase shifts at all energies. On the other hand, potentials with an r^{-2} singularity satisfy a generalised Levinson theorem (Swan 1963) and may be phase equivalent in spite of different numbers of bound states. This property can be employed to clarify (Baye 1987) the longstanding problem of the relations between deep and shallow nucleus–nucleus potentials (see references in Michel and Reidemeister (1985) and Baye (1987)). An exact phase equivalence can indeed be derived between a deep potential and a singular shallow potential.

   The aim of the present paper is to analyse systematically the possible types of phase-equivalent potentials which can be constructed with combinations of Darboux transformations of the radial Schrödinger equation. Let us stress that our point of view is complementary to Sukumar's. In Sukumar (1985b), emphasis is laid on constructing 'non-singular' potentials (i.e. without any r^{-2} singularity other that the usual $l(l+1)r^{-2}$ centrifugal term). In fact, this is phrased as 'keeping the orbital momentum l constant' but this terminology is confusing since Darboux transformations of a radial equation cannot modify l. Of course, the restriction to non-singular potentials limits the study of phase equivalence to potentials with the same number of bound states. Here we adopt a different approach: we construct phase-equivalent potentials whose spectra are allowed to differ at most by one bound state and we study their

properties and wavefunctions. Three cases are considered: (i) suppression of the ground state, (ii) addition of a new ground state and (iii) no alteration of the spectrum. Case (iii) does not require the introduction of a singularity and is already discussed in § 4.3 of Sukumar (1985b).

In § 2, the construction of supersymmetric partners of the radial Hamiltonian is summarised. Cases (i), (ii) and (iii) are discussed in turn in § 3 and are illustrated by an example in § 4. Concluding remarks are presented in § 5.

2. Darboux transformations of the radial equation

The radial differential equation of the lth partial wave is

$$H\psi = (-d^2/dr^2 + V(r))\psi = E\psi \tag{2.1}$$

where units are chosen so that $\hbar^2/2m$ is unity. The real local potential V may include a Coulomb component, i.e. it may tend towards zero at infinity as r^{-1}; V also includes the $l(l+1)r^{-2}$ centrifugal term. The notations in (2.1) do not refer explicitly to l since the orbital momentum is not modified by the Darboux transformation and since l plays almost no role in the following. The potential V is allowed to be singular in some sense; it is bounded for $r > 0$ but behaves for small r values as

$$\lim_{r\to 0} V \sim n(n+1)r^{-2} \tag{2.2}$$

where n is a positive integer. Notice that this is a rather peculiar singularity since V is positive near $r = 0$ and the roots of the indicial equation are the integers $n+1$ and $-n$. In the usual terminology, V is regular if $n = l$ and singular if $n \neq l$.

For a given E, a solution ψ of (2.1) is physical if it remains bounded everywhere and satisfies

$$\lim_{r\to 0} \psi = 0. \tag{2.3}$$

We reserve the notations E and ψ to physical eigenvalues and eigenfunctions. For an arbitrary $\mathscr{E}$ replacing E, solutions of the differential equation (2.1) will be denoted as φ. If

$$\mathscr{E} = -\gamma^2 \qquad \gamma > 0 \tag{2.4}$$

is not a physical eigenvalue, we shall need the solution χ regular at the origin such that

$$\lim_{r\to 0} \chi \sim r^{n+1}/(2n+1)!!$$

$$\lim_{r\to\infty} \chi \sim \exp(\gamma r). \tag{2.5}$$

From χ, we can define a second solution

$$f = \chi \int_r^x (\chi(t))^{-2}\, dt \tag{2.6}$$

with the properties

$$\lim_{r\to 0} f \sim (2n-1)!!\, r^{-n} \tag{2.7a}$$

$$\lim_{r\to\infty} f \sim \exp(-\gamma r). \tag{2.7b}$$

Following Sukumar (1985a, b), let us factorise H as

$$H = A^+ A^- + \mathscr{E} \tag{2.8}$$

where $\mathscr{E}$ is the factorisation energy. The operators A^+ and A^- are defined by

$$A^- = (A^+)^+ = -\mathrm{d}/\mathrm{d}r + \mathrm{d}(\ln \varphi)/\mathrm{d}r \tag{2.9}$$

where φ is a solution of the differential equation $H\varphi = \mathscr{E}\varphi$. Notice that φ must be nodeless in order that A^+ and A^- be bounded. Hence $\mathscr{E}$ must be less than or equal to the ground-state energy $E^{(0)}$ of H. In the latter case, φ must be the ground-state wavefunction $\psi^{(0)}$ since other solutions possess a node. In the former case, φ can be proportional to χ, to f or to a nodeless combination of both. Then a sypersymmetric partner H_1 of H (or the Darboux-transformed operator H_1) can be defined as

$$H_1 = A^- A^+ + \mathscr{E}.$$

The potential defined by H_1 is

$$V_1 = V + \Delta V = V - 2\,\mathrm{d}^2(\ln \varphi)/\mathrm{d}r^2. \tag{2.10}$$

The behaviour of ΔV near $r = 0$ depends on the choice of φ but it is readily shown from (2.5) and (2.7) that any Darboux transformation introduces a singularity in the potential, except possibly for $n = 0$. For r large, ΔV decreases as r^{-2} but we shall not need this property in the following. To each wavefunction ψ of H corresponds a wavefunction ψ_1 of H_1 given by

$$\psi_1 = \exp(\mathrm{i}\phi)(E - \mathscr{E})^{-1/2} A^- \psi \tag{2.11}$$

except if $A^- \psi$ vanishes or does not satisfy (2.3). The spectra of H and H_1 are then identical except possibly for their ground states. Convenient choices for the phase ϕ are discussed below. Let us now consider in turn the different types of Darboux transformation.

If $\mathscr{E} = E^{(0)} = -\gamma_0^2$ and $\varphi = \psi^{(0)}$ (Sukumar's transformation T_1), the ground state is suppressed since $A^- \psi^{(0)} = 0$. The singularity of the new potential is given by

$$\lim_{r \to 0} V_1 \sim (n+1)(n+2)r^{-2}. \tag{2.12}$$

The wavefunctions of H_1 are given by (2.11) where we choose $\phi = 0$. Let us apply the asymptotic form of A^- to the asymptotic form of a scattering wavefunction (with k, η and δ as wavenumber, Sommerfeld parameter and phase shift respectively). One obtains

$$\lim_{r \to \infty} A^- \sin(kr - \tfrac{1}{2}l\pi - \eta \ln 2kr + \delta)$$

$$\sim \sin[kr - \tfrac{1}{2}l\pi - \eta \ln 2kr + \delta + \tan^{-1}(k/\gamma_0)]$$

which shows that the phase shift is increased by $\tan^{-1}(k/\gamma_0)$. It is convenient to transform $\varphi'\psi - \varphi\psi'$ in (2.11) with the help of the differential equation (2.1) to rewrite ψ_1 as

$$\psi_1 = (E - E^{(0)})^{1/2}[\psi^{(0)}]^{-1} \int_0^r \psi^{(0)}\psi \, \mathrm{d}t. \tag{2.13}$$

The behaviour of ψ_1 near $r = 0$ is then given immediately by

$$\lim_{r \to 0} \psi_1 \sim r^{n+2} \tag{2.14}$$

in agreement with (2.12).

If $\mathscr{E} < E^{(0)}$, the transformation T_2 adds a new ground state at energy $\mathscr{E}$. The function φ can be defined as

$$\varphi = [\alpha(\alpha+1)]^{-1/2}\tilde{\varphi}\left(\alpha + \int_r^x \tilde{\varphi}^{-2}\,dt\right) \tag{2.15}$$

where $\tilde{\varphi}$ is any solution of $H\tilde{\varphi} = \mathscr{E}\tilde{\varphi}$ whose inverse is square integrable and normalised to unity. The function φ is nodeless if

$$\alpha(\alpha+1) > 0 \tag{2.16}$$

since $\int_r^x \tilde{\varphi}^{-2}\,dt$ lies between 0 and 1. Its inverse is also square integrable and normalised to unity. Another useful definition of φ can be given in terms of a solution χ at energy $\mathscr{E}$ with the properties (2.5). One has

$$\varphi = \alpha'^{-1/2}\chi\left(\alpha' + \int_r^x \chi^{-2}\,dt\right) \tag{2.17}$$

with the condition

$$\alpha' > 0 \tag{2.18}$$

since the integral can take any value between 0 and $+\infty$. Notice that these expressions for φ are related by $\chi = \tilde{\varphi}\int_0^r \tilde{\varphi}^{-2}\,dt$ and $\alpha' = \alpha(\alpha+1)^{-1}$. Other definitions of φ can be derived but we shall only need (2.15) and (2.17) in the following. The potential V_1 behaves near $r = 0$ as

$$\lim_{r\to 0} V_1 \sim (n-1)nr^{-2}. \tag{2.19}$$

The singularity is modified, except for $n = 0$. However, we shall see below that H and H_1 do not have the same spectrum for $n = 0$. The normalised ground-state wavefunction of H_1 is given by

$$\psi_1^{(0)} = \varphi^{-1}. \tag{2.20}$$

The other wavefunctions are given by (2.11) with the convenient choice $\phi = \pi$. Since φ behaves as $\exp(\gamma r)$ for large r values, the same reasoning as for the T_1 transformation shows that the phase shift is decreased by $\tan^{-1}(k/\gamma)$. The wavefunctions can be rewritten as

$$\psi_1 = -(E-\mathscr{E})^{1/2}\varphi^{-1}\left[\int_0^r \varphi\psi\,dt - \int_0^x \varphi\psi\left(\int_t^x \varphi^{-2}\,du\right)dt\right]. \tag{2.21}$$

The additional definite integral in the bracket is the integration constant appearing when evaluating $\varphi'\psi - \varphi\psi'$. With (2.21), one has, in agreement with (2.19)

$$\lim_{r\to 0} \psi_1 \sim r^n. \tag{2.22}$$

This relation shows that ψ_1 does not satisfy condition (2.3) for $n = 0$. Trying to add a bound state to a potential with $n = 0$ produces a non-singular potential V_1. However, the spectra of H and H_1 are different since the solution $A^-\psi$ at energy E is not a physical eigenstate of H_1. For the same reason, there is no simple relation between their phase shifts. The $n = 0$ case is therefore useless for the construction of phase-equivalent potentials but is an interesting example of supersymmetric partners whose spectra are not simply linked.

If $\varphi = \chi$ (transformation T_3), the spectra of H and H_1 are identical. The singularity of V_1 is given by (2.12). The wavefunctions are given by (2.11) with $\phi = \pi$. Here and in the next case, we choose the phase ϕ differently from Sukumar (1985b) for later convenience. Then

$$\psi_1 = -(E - \mathcal{E})^{1/2} \chi^{-1} \int_0^r \chi \psi \, dt \tag{2.23}$$

shows that ψ_1 behaves near $r = 0$ as in (2.14). The phase shift increases by $\tan^{-1}(k/\gamma)$. Except for small r values, T_3 behaves as a limit of T_2 for α' tending towards infinity in (2.17) or α tending towards -1 in (2.15).

If $\varphi = f$ (transformation T_4), the spectrum is also left in general unchanged by the transformation. The singularity of V_1 is given by (2.19). The wavefunction ψ_1 is given by (2.11) with $\phi = 0$ or by

$$\psi_1 = -(E - \mathcal{E})^{1/2} f^{-1} \int_r^x f \psi \, dt. \tag{2.24}$$

The phase shift decreases by $\tan^{-1}(k/\gamma)$. The behaviour for $r \to 0$ is given by (2.22). Notice that here also, the case $n = 0$ leads to a different spectrum. Except asymptotically, T_4 behaves as the limit of T_2 for α' tending towards 0 in (2.17).

The different types of Darboux transformations and their main properties are summarised in table 1. They depend on the choice of the separation energy and, if $\mathcal{E} < E^{(0)}$, on the choice of φ. The different transformations can be combined in two successive steps. Sukumar (1985b) has discussed the cases for which the singularity is not modified after the second step (remember that the second step corresponds to $n_1 = n \pm 1$). The non-trivial cases are then (T_1, T_4), (T_3, T_2) and (T_1, T_2). Here we focus on combinations which generate phase-equivalent potentials. The non-trivial cases are (T_1, T_3) (see Baye 1987), (T_4, T_2) and (T_1, T_2). We discuss them in turn in the next section.

Table 1. Properties of Darboux transformations.

	$\mathcal{E}$	φ	Action on the spectrum	Singularity modification	Phase shift modification
T_1	$E^{(0)}$	$\psi^{(0)}$	suppresses ground state	$2(n+1)r^{-2}$	$-\tan^{-1}(k/\gamma_0)$
T_2	$< E^{(0)}$	φ	adds new ground state[+]	$-2nr^{-2}$	$\tan^{-1}(k/\gamma)$
T_3	$< E^{(0)}$	χ	none	$2(n+1)r^{-2}$	$\tan^{-1}(k/\gamma)$
T_4	$< E^{(0)}$	f	none[+]	$-2nr^{-2}$	$-\tan^{-1}(k/\gamma)$

[+] If $n > 0$.

3. Phase-equivalent potentials

3.1. Suppression of the ground state

With T_1, the ground state is suppressed and the phase shift is modified by $-\tan^{-1}(k/\gamma_0)$ (see table 1). This phase shift change can be compensated by performing a transformation T_3 with $\mathcal{E} = E^{(0)}$. Notice that T_3 must follow T_1 since there must be no physical

state at energy $\mathscr{E}$. The combined transformation leads to a potential V_2 with singularity

$$\lim_{r \to 0} V_2 \sim (n+2)(n+3)r^{-2}. \tag{3.1}$$

This behaviour is a natural consequence of the generalised Levinson theorem (Swan 1963) as shown in Baye (1987).

This two-step process can be reduced to a single one. The function χ_1 of the second transformation is the solution, bounded near $r = 0$, of (see (2.10))

$$H_1 \chi_1 = [H - 2(\mathrm{d}^2/\mathrm{d}r^2) \ln \psi^{(0)}] \chi_1 = E^{(0)} \chi_1.$$

It can be expressed as

$$\chi_1 = [\psi^{(0)}]^{-1} \int_0^r [\psi^{(0)}]^2 \, \mathrm{d}t. \tag{3.2}$$

Combining the two potential modifications $-2(\mathrm{d}^2/\mathrm{d}r^2) \ln \psi^{(0)}$ and $-2(\mathrm{d}^2/\mathrm{d}r^2) \ln \chi_1$ (from (2.10)) yields

$$V_2 = V - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln \int_0^r [\psi^{(0)}]^2 \, \mathrm{d}t. \tag{3.3}$$

The potential V_2 can thus be obtained in a single step from the ground-state wavefunction of H. With (3.3), it is easy to prove (3.1) directly. Also (3.3) shows that, for large r values,

$$V_2 \sim V - 2\frac{\mathrm{d}}{\mathrm{d}r} [\psi^{(0)}]^2 \tag{3.4}$$

so that the two potentials do not differ significantly as soon as $\psi^{(0)}$ becomes small.

The wavefunctions ψ_2 of H_2 can also be obtained directly from the wavefunctions of H. Let us combine (2.13) for the first step T_1 with (2.11) for the second step T_3. Our choice for the phase ϕ and (3.2) then lead to

$$\psi_2 = \psi - \psi^{(0)} \int_0^r \psi^{(0)} \psi \, \mathrm{d}t \left(\int_0^r [\psi^{(0)}]^2 \, \mathrm{d}t \right)^{-1}. \tag{3.5}$$

Clearly, the wavefunctions ψ_2 and ψ have the same asymptotic behaviour as soon as $\psi^{(0)}$ is negligible. In particular, for positive energies, ψ_2 and ψ provide the same phase shifts. Equation (3.5) also displays the fact that no wavefunction of H_2 corresponds to $\psi^{(0)}$. Finally, a direct calculation shows that for bound states ψ_2 and ψ have the same normalisation.

3.2. Addition of a new ground state

With T_2, a new bound state is added below the ground state, at energy $\mathscr{E}$, but the phase shift is modified by $\tan^{-1}(k/\gamma)$. This phase shift change can be corrected with a transformation T_4 involving the same separation energy. However, the fact that $\mathscr{E}$ must be less than the ground-state energy implies that T_4 must precede T_2. The singularity modifications (see table 1) then lead to

$$\lim_{r \to 0} V_2 \sim (n-2)(n-1)r^{-2} \qquad n > 0 \tag{3.6}$$

in agreement with the generalised Levinson theorem. However, the n values for which (3.6) is valid and V_2 is phase equivalent to V require a closer examination.

The first transformation T_4 is performed with a function f, which is the solution, bounded at infinity, of $Hf = \mathscr{E}f$. The function ψ_1 appearing in the second transformation T_2 is easily expressed as a function of f

$$\varphi_1 = f^{-1}\left(\alpha + \int_r^\infty f^2\,\mathrm{d}t\right) \qquad \alpha > 0 \tag{3.7}$$

where we have chosen $\chi = f^{-1}$ in (2.17). The definition of the positive parameter α is unambiguous if the normalisation of f is defined (see (2.7a)). The potential resulting from the combined transformation is

$$V_2 = V - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\alpha + \int_r^\infty f^2\,\mathrm{d}t\right). \tag{3.8}$$

With (2.7) and (3.8) it is easy to prove that (3.6) holds for $n > 0$ and that V_2 and V have the same behaviour for large r values since f decreases exponentially.

With (2.24) for T_4, (2.11) for T_2 and (3.7), one obtains the wavefunctions of H_2 as

$$\psi_2 = \psi - f\int_r^\infty f\psi\,\mathrm{d}t\left(\alpha + \int_r^x f^2\,\mathrm{d}t\right)^{-1}. \tag{3.9}$$

This expression shows that ψ_2 and ψ have the same asymptotic behaviour and therefore provide for positive energies the same phase shifts. In addition, the wavefunction of the new ground state is given by (2.20) as

$$\psi_2^{(0)} = \alpha^{1/2}f\left(\alpha + \int_r^x f^2\,\mathrm{d}t\right)^{-1}. \tag{3.10}$$

It is readily verified by direct calculation that this wavefunction is orthogonal to the other ones and that the different bound-state wavefunctions are normalised.

Equations (3.9) and (3.10) provide the conditions at which a bound state can be added to a given potential. Indeed, let us consider the behaviour of any wavefunction ψ_2 (including $\psi_2^{(0)}$) at small r values. Since

$$\int_0^x f\psi\,\mathrm{d}t < \infty$$

one has with (2.7a)

$$\lim_{r\to 0}\psi_2 \sim r^{n-1} \qquad n > 0$$
$$\sim 1 \qquad n = 0. \tag{3.11}$$

The functions ψ_2 do not satisfy the condition (2.3) for $n = 0$ or 1. In these cases, the potential V_2 calculated with (3.8) does not present an r^{-2} singular behaviour. However, its physical eigenfunctions are not given by (3.9) and (3.10) and appear at other energies. The potential has a bound spectrum which differs from the spectrum of V and provides different phase shifts. The contents of the present paragraph are therefore restricted to potentials whose index n is greater than 1. If $n > 1$, phase-equivalent potentials depending on two free parameters $\mathscr{E} < E^{(0)}$ and $\alpha > 0$ can be constructed.

3.3. Unchanged spectrum

Table 1 indicates that several combinations of Darboux transformations lead to phase-equivalent potentials with an unchanged spectrum if they make use of the same

separation energy $\mathscr{E}$. However, it is easily shown that the combinations (T_3, T_4) and (T_4, T_3) do not modify the potential. The only interesting case is therefore to suppress the ground state with T_1 and to reintroduce it at the same energy with T_2 (Sukumar 1985b). No singularity is introduced in this process. The potential V_2 differs from V because of the existence of a free parameter in T_2. Notice that T_2 followed by T_1 would not modify the potential since the added ground state would be suppressed with the inverse transformation for any choice of the parameters $\mathscr{E}$ and α.

The first transformation T_1 makes use of $\psi^{(0)}$. For the second transformation T_2, we choose $\varphi = [\psi^{(0)}]^{-1}$ in (2.15) so that

$$\varphi_1 = [\psi^{(0)}]^{-1} \left(\alpha + \int_r^x [\psi^{(0)}]^2 \, dt \right) \tag{3.12}$$

with the condition (2.16) (α^{-1} is Sukumar's λ). The new potential is

$$V_2 = V - 2 \frac{d^2}{dr^2} \ln \left| \alpha + \int_r^x [\psi^{(0)}]^2 \, dt \right|. \tag{3.13}$$

Notice the absolute value, which is necessary if $\alpha < -1$. Again, with (3.13), one readily shows that V_2 behaves near $r = 0$ as in (2.2) and does not differ from V when $\psi^{(0)}$ becomes negligible.

With (2.13), (2.11), (2.20) and (3.12), one obtains

$$\psi_2 = \psi - \psi^{(0)} \int_r^x \psi^{(0)} \psi \, dt \left(\alpha + \int_r^x [\psi^{(0)}]^2 \, dt \right)^{-1} \tag{3.14}$$

for $\psi \neq \psi^{(0)}$ and

$$\psi_2^{(0)} = [\alpha(\alpha+1)]^{1/2} \psi^{(0)} \left(\alpha + \int_r^x [\psi^{(0)}]^2 \, dt \right)^{-1}. \tag{3.15}$$

Notice that (3.15) is not a particular case of (3.14) for a question of normalisation. The wavefunctions ψ_2 and ψ differ by a short-ranged term and provide the same phase shifts at positive energies. For small r values the second term in the expression (3.14) of ψ_2 is smaller than the first one because of the orthogonality between $\psi^{(0)}$ and ψ. The functions ψ_2 and ψ present the same r^{n+1} behaviour in agreement with the properties of the potential.

Finally, it is worthwhile discussing here the limiting cases $\alpha = 0$ and $\alpha = -1$. For $\alpha = 0$, (3.12) and (2.7) show that φ_1 is in fact a function f_1. The transformation T_2 becomes T_4 and table 1 indicates that the phase shift is modified by the combined transformation. We shall exemplify this point in § 4. For $\alpha = -1$, φ_1 becomes χ_1 (see (2.5)), T_2 becomes T_3 and one recovers the case discussed in § 3.1 where the ground state is suppressed. This property is also illustrated in § 4.

4. An illustrative example

Here we illustrate some of the properties discussed in § 3 and clarify their interpretation with a realistic potential encountered in nucleus-nucleus collisions. We choose the simple potential of Buck *et al* (1977) which reproduces accurately the $l = 0\text{-}6$ phase shifts of $\alpha + \alpha$ scattering between 0 and 40 MeV in the centre of mass frame. For $l = 0$, this potential is given in MeV, for r in fm, by

$$V(r) = -122.6225 \exp(-0.22r^2) + 4e^2 \operatorname{erf}(0.75r)/r. \tag{4.1}$$

The other *l* values are obtained by adding the usual centrifugal potential with $\hbar^2/2m =$ 10.368 MeV fm^2. The corresponding Hamiltonian possesses two bound states with $l = 0$ and one with $l = 2$ (odd partial waves are unphysical because of the identity of the colliding bosons).

The $l = 0$ potential is illustrated by the lower broken curves in figure 1; the energies of its bound states are also displayed. Suppressing the ground state as in § 3.1 leads to the phase-equivalent potential represented by the upper broken curve on the RHS of figure 1. (The labelling -1 of this curve is explained below.) Iterating the procedure to suppress the excited state provides a potential which is discussed in Baye (1987).
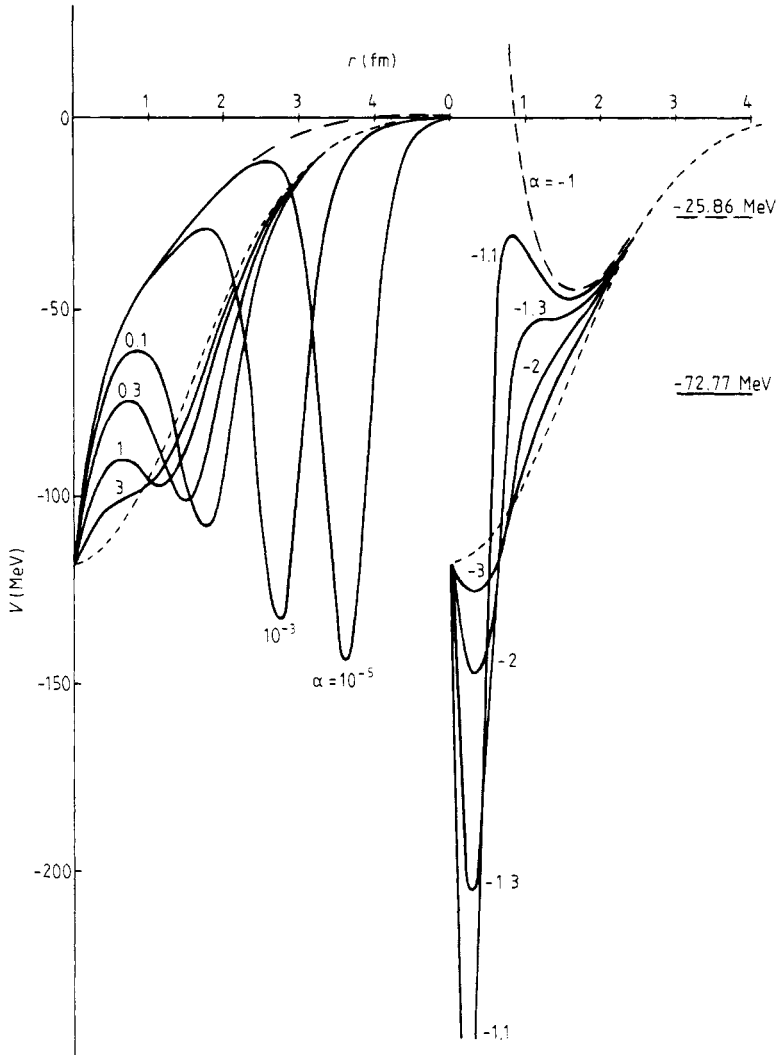

Figure 1. The $l = 0$ potential (- - -) of equation (4.1) and the phase-equivalent potentials (——) for different values of α (equation (3.13)) plotted for $\alpha > 0$ on LHS and $\alpha < -1$ on RHS. The upper envelopes of the potentials are also shown (— —) coinciding on the RHS with the phase-equivalent potential (3.3) for $\alpha = -1$. The bound state energies of the $l = 0$ potential are indicated at -25.86 MeV and -72.77 MeV.

Trying to add a bound state to potential (4.1) fails in agreement with the discussion in § 3.2; the resulting potential provides a different bound spectrum and different phase shifts. Now we make use of potential (4.1) to clarify the properties of phase-equivalent potentials with an unchanged spectrum (§ 3.3). The phase-equivalent potentials are calculated numerically from (3.13). The wavefunction $\psi^{(0)}$ is computed with a standard Numerov technique (Raynal 1972) at equally spaced mesh points with a step h. The integral $\int_0^r [\psi^{(0)}]^2 \, dt$ is determined at each mesh point with the Adams 'interpolation' formula (Abramowitz and Stegun 1972, equation 25.5.5). The second derivative is calculated with a five-point differentiation formula. For $h = 0.01$, the energies of the different phase-equivalent potentials reproduce those corresponding to (4.1) with an accuracy better than 10^{-3} MeV. The accuracy of the phase shifts is better than 10^{-4} up to 200 MeV.

Potentials obtained with (3.13) for positive values of parameter α are presented as full curves in the LHS of figure 1. For a large α value, the new potential V_2 does not differ much from V (lower broken curve) since $\int_r^\infty [\psi^{(0)}]^2 \, dt$ is bounded by unity. The difference between both potentials increases when α decreases. For any $\alpha > 0$, V_2 is equal to V for $r = 0$ and larger than V for small r values owing to the fact that the logarithm in (3.13) reaches its maximum $\ln(\alpha + 1)$ at $r = 0$ with an r^{2n+3} behaviour. On the other hand, for large r values, the logarithm tends exponentially towards its minimum $\ln \alpha$ so that V_2 tends exponentially towards V from below. Hence the continuous potential V_2 is equal to V at some intermediate point. When α becomes small, a second well appears whose minimum moves towards large r values as the logarithm of α. The ground-state wavefunction becomes essentially concentrated in this well while the first excited state remains located at smaller distances. The node in the excited wavefunction which ensures the orthogonality with the ground state follows the delocalisation of the second well. For $\alpha = 0$, the ground-state well disappears and the excited state becomes the (nodeless) ground state of a potential which is the upper envelope of the different potentials V_2. This limit potential is not phase equivalent to V in agreement with the Levinson theorem.

Potentials corresponding to α values less than -1 are displayed as full curves in the RHS of figure 1. The relative locations of V and V_2 are interchanged with respect to $\alpha > 0$. The potentials do not differ much when $|\alpha|$ is large, as expected. When α tends towards -1, the ground state is progressively squeezed in a deep narrow well located near $r = 0$. At the limit $\alpha = -1$, this well disappears and the potential becomes the upper envelope of the different potentials V_2. Here, however, the limit potential remains phase equivalent by becoming singular. This potential is obtained directly with the ground-state suppression.

The $l = 2$ potential, with its $6r^{-2}$ singularity can be employed to illustrate the addition of a bound state as shown in figure 2. This potential possesses a single bound state at -22.28 MeV. The numerical technique is similar to the one described above but the function f appearing in (3.8) is calculated backwards. The accuracy of the results is poorer than in the preceding case. For the smaller step $h = 0.0025$, the energy $\mathscr{E}$ of the added ground state is obtained to an accuracy of 10^{-3} MeV but the energy of the former ground state is reproduced to only 10^{-2} MeV. The accuracy of the phase shifts is better than 2×10^{-3} up to 200 MeV. The potential V_2 varies strongly at small r values since $\lim_{r \to 0} \int_r^\infty f^2 \, dt$ diverges. A cutoff of the potential for these r values stabilises the numerical computation without modifying the energies and phase shifts. Adding a bound state introduces two degrees of freedom. As an example, we have added a bound state at -50 MeV or at -100 MeV for three values of the parameter α. The new
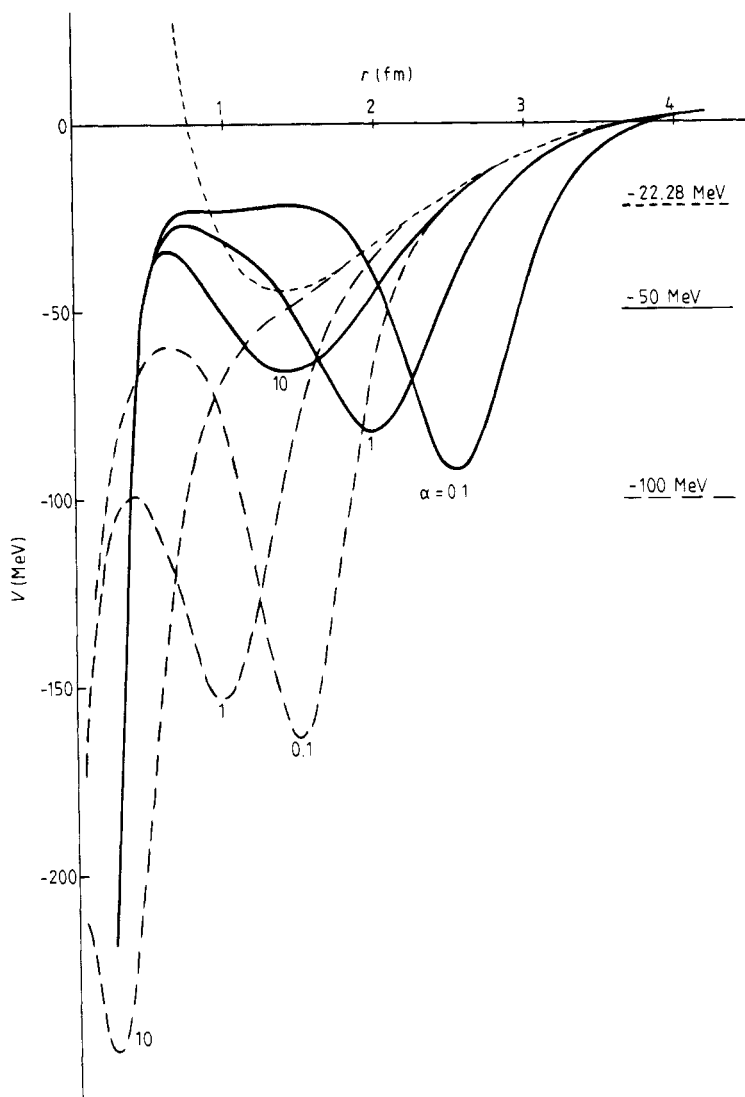
Figure 2. The $l = 2$ potential with a single bound state at $E^{(0)} = -22.28$ MeV (- - -) and the phase-equivalent potentials for the addition of a bound state at $\mathscr{E} = -50$ MeV (———) or $\mathscr{E} = -100$ MeV (— —) for different values of α (equation (3.8)).

ground state is essentially localised in a rather narrow well whose minimum varies logarithmically with α. As expected from (3.8), V_2 tends faster towards V for large r values if $|\mathscr{E}|$ or α increases.

5. Conclusions

Supersymmetric quantum mechanics allows one to derive potentials which provide the same phase shifts as a given potential, even if the number of bound states is modified. However, adding a bound state or suppressing the ground state yields a modification

of the potential singularity at the origin in agreement with a generalised Levinson theorem (Swan 1963). In particular, the addition of a new ground state requires that the potential be already singular. Otherwise, the potential obtained from the Darboux transformation leads to a different spectrum and different phase shifts.

From a numerical point of view, the phase-equivalent potentials arising from suppressing the ground state or leaving the spectrum unchanged are determined to an excellent accuracy without very elaborate procedures. On the other hand, adding a bound state is far less accurate and, for small r values, is unstable.

Most of the phase-equivalent potentials presented as examples display a rather complicated shape. They present additional maxima and minima which are well explained by the properties of the potential modification ΔV. Only the ground-state suppression provides simply shaped curves. Therefore we think this case should have the most useful applications. For example, it resolves the ambiguities observed in the construction of accurate $\alpha + \alpha$ scattering potentials (Baye 1987).

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